Images of multilinear polynomials on upper triangular matrices

Joint work with Ivan Gonzales Gargate (arXiv:2106.12726)

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Motivations: traceless matrices

Exercise
If $A, B \in M_k(F)$. Then $\text{tr}([A, B]) = 0$.

Theorem (Shoda-Albert-Muckenhoupt)
Let $M \in M_k(F)$ such that $tr(M) = 0$. Then there exists $A, B \in M_k(F)$ such that $M = [A, B]$.

In other words
The map

$$\begin{array}{ccc} M_k(F) \times M_k(F) & \longrightarrow & M_k(F) \\ (A, B) & \longmapsto & [A, B] \end{array}$$

is surjective

Images of polynomials on algebras

If $f(x_1, \ldots, x_m) \in F\langle X \rangle$, then f defines a map

$$\begin{array}{rccc} f: & M_k(F)^m & \longrightarrow & M_k(F) \\ & (A_1, \ldots, A_m) & \longmapsto & f(A_1, \ldots, A_m) \end{array}$$

Question (Kaplansky):

Which subsets of $M_k(F)$ are image of some polynomial $f \in F\langle X \rangle$?

Examples

- If $f(x_1, \ldots, x_m) = x_1$, then $Im(f) = M_k(F)$.
- (Amitsur-Levitzki) If $St_{2k}(x_1, \ldots, x_{2k}) = \sum_{\sigma \in S_{2k}} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(2k)}$, then $Im(St_{2k}) = \{0\}$ in $M_k(F)$.
- If $f(x_1, x_2) = [x_1, x_2]$, then $\mathrm{Im}(f) = sl_k(F)$, the set of trace zero matrices (Shoda, Albert and Muckenhoupt).
- If $h(x_1, x_2, x_3, x_4) = [x_1, x_2] \circ [x_3, x_4]$, then the image of h em $M_2(F)$ is F (the set of scalar matrices).

The structure of the image of a polynomial

Proposition
If $f \in F\langle X \rangle$, then $\text{Im}(f)$ is invariant under conjugation by invertible matrices.

Proposition
If $f \in F\langle X \rangle$ is multilinear, then $\text{Im}(f)$ is invariant under scalar multiplication.

Question (Lvov)
Let f be a multilinear polynomial over a field F. Is the image of f on the matrix algebra $M_k(F)$ a vector space?

The linear span of the image of a polynomial

Proposition
Let $f \in F\langle X\rangle$. Then the linear span of $\mathrm{Im}(f)$ is a Lie Ideal in $M_k(F)$.

Theorem (Herstein)
If F is an infinite field, any Lie ideal of $M_k(F)$ is one of the following:

$$M_k(F), sl_k(F), F \text{ or } \{0\}.$$

Corollary
Let $f \in F\langle X\rangle$. Then the linear span of $\mathrm{Im}(f)$ is one of the following:

$$M_k(F), sl_k(F), F \text{ or } \{0\}.$$

Conjecture (Lvov-Kaplansky)
Let $f(x_1, \ldots, x_m) \in F\langle X\rangle$ be a multilinear polynomial. Then $\mathrm{Im}(f)$ in $M_k(F)$ is one of the following:

$$M_k(F), sl_k(F), F \text{ or } \{0\}.$$

Example

Example
Let $f(x) = x^k$. What is the image of f on $M_k(F), k \geq 2$?

- $f(E_{ii}) = E_{ii}$.
- $f(E_{ii} + E_{ij}) = E_{ii} + E_{ij}$, if $i \neq j$.
- As a consequence, $\text{span}(\text{Im}(f)) = M_k(F)$.
- If $E_{ij} \in \text{Im}(f)$, $i \neq j$, then $E_{ij} = A^k$, for some $A \in M_k(F)$.
- Since E_{ij} is nilpotent, so is A.
- Then $A^k = 0$, and $E_{ij} = 0$.
- In particular, $\text{Im}(f)$ is not a vector subspace of $M_k(F)$.

Some known results

Theorem (Kanel-Belov, Malev and Rowen, 2012 and Malev, 2014)
If f is a multilinear polynomial evaluated on the matrix ring $M_2(F)$, where F is a quadratically closed field, or $F = \mathbb{R}$, then $\mathrm{Im}(f)$ is one of the following:

$$M_2(F), sl_2(F), F \text{ or } \{0\}.$$

Theorem (Malev, 2014)
If f is a multilinear polynomial evaluated on the matrix ring $M_2(F)$ (where F is an arbitrary field), then $\mathrm{Im}(f)$ is either 0, F, or $sl_2 \subseteq \mathrm{Im}(f)$.

Some known results

Theorem (Kanel-Belov, Malev and Rowen, 2016)
Let F be an algebraically closed field. Then the image of a multilinear polynomial $f \in F\langle X \rangle$ evaluated on $M_3(F)$ is one of the following:

- $\{0\}$,
- F;
- a dense subset of $sl_3(F)$;
- a dense subset of $M_3(F)$;
- the set of 3-scalar matrices, or
- the set of scalars plus 3-scalar matrices.

The non-subspace ones have not been showed to be the image of any polynomial.

Some known results

The case $n = \infty$ has also a solution:

Theorem (D. Vitas)
Let V be an infinite-dimensional vector space over a division algebra D (over any field F). If $f(x_1, \ldots, x_m)$ is a multilinear polynomial, then the image of f on $A = End_D(V)$ is A.

Some known results

Theorem(Dykema and Klep, 2016)
Let $f \in \mathbb{C}\langle x_1, x_2, x_3 \rangle$ be a multilinear polynomial of degree three.

1. If $k \in \mathbb{N}$ is even, then the image of f in $M_k(\mathbb{C})$ is a vector space.
2. If $k \in \mathbb{N}$ is odd and $k < 17$, then the image of f in $M_k(\mathbb{C})$ is a vector space.

The Mesyan Conjecture

Proposition
Let $m, n \geq 2$ and $f(x_1, \ldots, x_m) \in K\langle x_1, \ldots, x_m \rangle$ be a multilinear polynomial. If $n \geq \frac{m+1}{2}$, then the linear span of the image of f contains $sl_n(K)$.

Conjecture
Let $m, n \geq 2$ and $f(x_1, \ldots, x_m) \in K\langle x_1, \ldots, x_m \rangle$ be a multilinear polynomial. If $n \geq \frac{m+1}{2}$, then the image of f contains $sl_n(K)$.

- This is a weaker version of the Lvov-Kaplansky Conjecture.
- Solutions are known for $m \leq 4$.
- A complete solution is known for the algebra of finitary matrices.

Commutators and images of polynomials

Theorem (M. Bresar)
Let A be a unital algebra over an infinite field F, and let $f \in K\langle X\rangle$. The following two statements are equivalent:

1. f is neither an identity nor a central polynomial of any nonzero homomorphic image of A.
2. $[A, A] \subseteq \mathrm{span}(f(A))$ and A is equal to its commutator ideal.

Problem

Let A be an associative algebra (or Lie, Jordan, or some algebra in your favorite variety) and let $f(x_1, \ldots, x_m)$ be a multilinear polynomial (multilinear in the free algebra of the considered variety). Is the image of f is a vector subspace of A?

Variations in the Lvov-Kaplansky Conjecture - Some results

Theorem (Kanel-Belov, Malev, Rowen, 2017)
For any algebraically closed field F of characteristic $\neq 2$, the image of any Lie polynomial f (not necessarily homogeneous) evaluated on $sl_2(F)$ is either $sl_2(F)$, or 0, or the set of trace zero non-nilpotent matrices.

Theorem (Špenko, 2012, Anzis, Emrich and Valiveti, 2015)
The image of multilinear Lie polynomials of degree ≤ 4 on sl_k, $su(k)$ and $so(k)$ are vector subspaces.

Theorem (Ma and Oliva, 2016)
The image of any degree-three multilinear Jordan polynomial over the Jordan subalgebra of symmetric elements in $M_k(F)$ is a vector space.

Theorem (Malev, Pines, 2020)
The image of a (nonassociative) multilinear polynomial evaluated on the rock-paper-scissors algebra is a vector subspace.

Variations in the Lvov-Kaplansky Conjecture - Some results

Theorem (Malev, 2021)
If p is a multilinear polynomial evaluated on the quaternion algebra $\mathbb{H}$, then Imp is either 0, or $\mathbb{R} \subseteq \mathbb{H}$ (the space of scalar quaternions), or V (the space of vector quaternions), or $\mathbb{H}$.

Theorem (Belov, Malev, Pines and Rowen, 2022)
The image of a multilinear polynomial p on an Octonion algebra $\mathbb{O}$ is either $\{0\}$, F, V or $\mathbb{O}$.

Variations in the Lvov-Kaplansky Conjecture - Some results

Let now J_n be the Jordan $\mathbb{R}-$algebra with basis $\{e_0 = 1, e_1, \ldots, e_{n-1}\}$ and multiplication table $e_i \circ e_j = \delta_{ij}$.

Theorem (Malev, Yavich and Shayer, 2021)
Let p be any commutative non-associative polynomial. Then its evaluation on the algebra J_n is either $\{0\}$, or $\mathbb{R}$ (i.e. one-dimensional subspace spanned by the identity element), the space V of pure elements (the $(n-1)$-dimensional vector space spanned by $e_1, \ldots e_{n-1}$), or J_n.

Variations in the Lvov-Kaplansky Conjecture - General statement

Generalized Lvov-Kaplansky Conecture
Let F be a field and let $\mathcal{V}$ be a variety of F-algebras. If $f(x_1, \ldots, x_m)$ is a polynomial in the free $\mathcal{V}$-algebra, and A is a finite dimensional simple algebra in $\mathcal{V}$, then the image of f on A is a vector space.

Theorem (Fagundes, de Mello, 2018)
Let f be a multilinear polynomial of degree ≤ 4. Then the image of f on $UT_k(F)$ is UT_k, J or J^2.

Theorem (Fagundes, de Mello, 2018)
If f is a multilinear polynomial, then the linear span of $\mathrm{Im}(f)$ on $UT_k(F)$ is $UT_k(F)$ or J^r, for some $r \geq 0$.

Theorem (Fagundes, 2018)
Let $k \geq 2$ and $m \geq 1$ be integers. Let $f(x_1, \ldots, x_m) \in F\langle X \rangle$ be a nonzero multilinear polynomial. Then the image of f on strictly upper triangular matrices is either 0 or J^m.

Image of multilinear polynomials on UT_n - the solution

The commutator-degree of polynomials

We have a strictly descending chain of T-ideals of $F\langle X\rangle$:

$$F\langle X\rangle \supsetneq \langle [x_1,x_2]\rangle^T \supsetneq \langle [x_1,x_2][x_3,x_4]\rangle^T \supsetneq \langle [x_1,x_2][x_3,x_4][x_5,x_6]\rangle^T \supsetneq \cdots$$

We say that f has *commutator-degree* r if
$f \in \langle [x_1,x_2][x_3,x_4]\cdots[x_{2r-1},x_{2r}]\rangle^T$ and
$f \notin \langle [x_1,x_2][x_3,x_4]\cdots[x_{2r+1},x_{2r+2}]\rangle^T$.

Proposition
Let $f \in F\langle X\rangle$. Then f has commutator-degree r if and only if
$f \in Id(UT_r)$ and $f \notin Id(UT_{r+1})$.

The commutator-degree of polynomials

Lemma
Let A be a unitary algebra over F and let

$$f(x_1,\ldots,x_m)=\sum_{\sigma\in S_n}\alpha_\sigma x_{\sigma(1)}\cdots x_{\sigma(m)}.$$

be a multilinear polynomial in $F\langle X\rangle$.

1. If $\sum_{\sigma\in S_n}\alpha_\sigma\neq 0$, then $Im(f)=A$.
2. $f\in\langle[x_1,x_2]\rangle^T$ if and only if $\sum_{\sigma\in S_n}\alpha_\sigma=0$.

Remark
The above characterizes the multilinear polynomials with commutator degree 0.

Suitable sets of permutations

If $k \geq 1$, let $T_1, \ldots, T_k \subseteq \{1, \ldots, m\}$ and $1 \leq t_1 < \cdots < t_k$ such that

$$\{1, \ldots, m\} = T_1 \dot{\cup} \cdots \dot{\cup} T_k \dot{\cup} \{t_1, \ldots, t_k\}$$

and let us denote by $S(k, T, t)$ the subset of S_m consisting of all permutations σ satisfying:

- $\sigma(\{1, 2, \cdots, h_1 - 1\}) = T_1$
- $\sigma(h_1) = t_1$
- if $i \in \{2, \ldots, k\}$, $\sigma(\{h_1 + \cdots + h_{i-1} + 1, \ldots, h_1 + \cdots + h_i - 1\}) = T_i$
- if $i \in \{2, \ldots, k\}$, $\sigma(h_1 + \cdots + h_i) = t_i$

where $h_i = |T_i| + 1$.

Suitable sums of coefficients

And now consider the following sum of coefficients of f:

$$\beta^{(k,T,t)} = \sum_{\sigma \in S(k,T,t)} \alpha_\sigma$$

Theorem

Let $f(x_1, \ldots, x_m) = \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(m)}$ be a multilinear polynomial in $F\langle X \rangle$. Then the following assertions are equivalent.

1. The polynomial f has commutator-degree $r \geq 1$
2. For all $k < r$, and for any T, and t, we have $\beta^{(k,T,t)} = 0$ and there exist T, and t such that $\beta^{(r,T,t)} \neq 0$

The proof

Lemma
Let $f(x_1, \dots, x_m) = \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(m)}$ be a multilinear polynomial in $F\langle X\rangle$. If $\beta^{(k,T,t)} = 0$, for all $k < n$, and for any T, and t, then $f \in Id(UT_n)$.

Lemma
Let $f(x_1, \dots, x_m) = \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(m)}$ be a multilinear polynomial in$F\langle X\rangle$ and let $n > 0$. If there exist T, and t such that $\beta^{(n,T,t)} \neq 0$, then $f \notin Id(UT_{n+1})$.

The proof

Theorem
Let $f(x_1, \ldots, x_m) = \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(m)}$ be a multilinear polynomial in $F\langle X \rangle$. Then the following assertions are equivalent.

1. The polynomial f has commutator-degree $r \geq 1$
2. $f \in Id(UT_r) \setminus Id(UT_{r+1})$
3. For all $k < r$, and for any T, and t, we have $\beta^{(k,T,t)} = 0$ and there exist T, and t such that $\beta^{(r,T,t)} \neq 0$

The Lvov-Kaplansky conjucture for UT_n is true

Theorem (I. Gonzales, T. C. de Mello, 2021)
Let $f \in F\langle X\rangle$ be a multilinear polynomial. Then the image of f on $UT_n(F)$ is J^r if and only if f has commutator-degree r.

Remark
The above theorem was also proved by Luo and Wang (2022) using different methods. Their proof holds also for (big enough) finite fields.

Related problems

Let us consider the following descending chain of T-ideals in $F\langle X\rangle$

$$F\langle X\rangle \supsetneqq \langle St_2\rangle^T \supsetneqq \langle St_4\rangle^T \supsetneqq \langle St_6\rangle^T \supsetneqq \cdots$$

Define a polynomial f to be of *St-degree* k if

$$f \in \langle St_{2k}\rangle^T \text{ and } f \notin \langle St_{2(k+1)}\rangle^T.$$

Problem
Characterize multilinear polynomials of St-degree k by means of its coefficients.

Other variations - graded polynomials

Let $A = \bigoplus_{g \in G} A_g$ be a G-graded F-algebra and let $f(x_1, \ldots, x_m) \in F\langle X|G\rangle$ be a graded multilinear polynomial. Is the image of f in A a vector space?

Thank you!